Higher approximations in boundary-layer theory

Part 3. Parabola in uniform stream

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The classical laminar boundary layer on a parabolic cylinder is calculated using the Blasius series, with modifications to improve convergence, and supplemented by an asymptotic expansion valid far downstream from the nose. The flow due to displacement thickness is thereby found with sufficient accuracy to permit evaluation of its second-order effect upon the boundary layer near the stagnation point. The skin friction and heat transfer are found to be reduced there by both displacement and curvature.

1. Introduction

The second approximation for Prandtl's laminar boundary-layer theory was studied in Part 1 (Van Dyke 1962*a*) and applied to leading edges in Part 2 (Van Dyke 1962*b*). Thus the second-order solution has in principle been calculated for the stagnation point of a round-nosed plane or axisymmetric body, and also for a cusped leading edge at ideal incidence.

These results are necessarily incomplete, however, in that one of the second-order corrections is proportional to the change in inviscid surface speed induced at the nose by the displacement effect of the first-order boundary layer. This displacement speed is global in nature, depending on the entire course of the boundary layer. Therefore completion of the local solution at the nose can be undertaken only if the entire body shape is specified.

At present the displacement speed can be calculated in principle only for unseparated flow. It is known in closed form for the semi-infinite flat plate (for which it vanishes) and the wedge (Kaplun 1954). It has been calculated approximately for the finite flat plate (cf. Part 2) by Kuo (1953). Interest in these cases is diminished, however, by the fact that the boundary-layer approximation fails in the vicinity of a sharp leading edge.

We study here what is probably the simplest case of a round-nosed body: the parabolic cylinder in a uniform incompressible stream. The first-order boundary-layer solution is sought by series expansion, and much of the paper is devoted to manipulating the series to provide sufficient accuracy for the second approximation. We follow where possible the notation of previous parts, and refer to an equation in Part 1 or 2 by giving its number preceded by the Roman numeral I or II.

2. First-order boundary-layer solution

Consider a semi-infinite parabolic cylinder at zero incidence in a uniform stream. As in Part I, choose units such that the nose radius and free-stream speed are unity. Introduce parabolic co-ordinates ξ, η according to

$$x-\tfrac{1}{2}+iy = \tfrac{1}{2}(\xi+i\eta)^2 \tag{2.1a}$$

so that, as indicated in figure 1, the body is described by $\eta = 1$; then the length element dl is given by

$$dl^2 = (\xi^2+\eta^2)(d\xi^2+d\eta^2). \tag{2.1b}$$

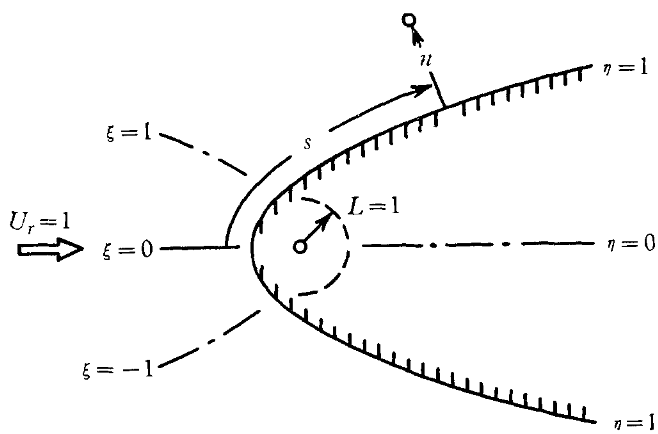

FIGURE 1. Notation for parabola.

The basic inviscid flow is well known, the stream function being

$$\Psi_1 = \xi(\eta-1) \tag{2.2}$$

and the surface speed, required for the boundary-layer calculation, is found to be

$$U_1(s,0) = \xi/(1+\xi^2)^{\frac{1}{2}}. \tag{2.3}$$

Integrating (2.1*b*) with $\eta = 1$ gives the curvilinear distance s along the surface as

$$s = \tfrac{1}{2}[\xi(1+\xi^2)^{\frac{1}{2}}+\sinh^{-1}\xi]. \tag{2.4}$$

2.1. *Blasius series for skin friction*

The boundary-layer solution cannot be calculated in closed form. We adopt the approximation of Blasius and Howarth (Schlichting 1960, p. 146), which yields an expansion in powers of s starting with Hiemenz's solution for the plane stagnation point.

The inviscid surface speed must be expanded in powers of s. This is most easily done by eliminating ξ between (2.3) and (2.4) to obtain

$$s = \tfrac{1}{2}[U_1/(1-U_1^2)+\tanh^{-1}U_1] \tag{2.5a}$$

and differentiating to find

$$dU_1/ds = (1-U_1^2)^2. \tag{2.5b}$$

Then substituting an assumed series for U_1 and equating like powers of s leads to the required result

$$U_1(s,0) = s - \frac{2}{3}s^3 + \frac{11}{15}s^5 - \frac{292}{315}s^7 + \frac{3548}{2835}s^9 - \frac{273{,}766}{155{,}925}s^{11} + \dots \quad (2.5c)$$

We note that this series converges only for $s \leqslant \frac{1}{4}\pi = 0{\cdot}7854$, the limitation arising from singularities at $\pm i\frac{1}{4}\pi$ in the plane of s regarded as a complex variable. According to (2.4) these correspond to singularities at $\pm i$ in the plane of ξ, which arise from the conformal mapping that generates the parabola, and are evident as the roots of the denominator in (2.3).

The formal Blasius series for the boundary-layer solution is now obtained at once from the general theory, for which the required universal functions have been computed with great accuracy by Tifford (1954). Substituting into equations (9.18) and (9.19) of Schlichting (1960) gives for the (dimensionless) stream function in the boundary layer

$$\begin{aligned}\psi_1 = {} & sf_1(N) - \tfrac{8}{3}s^3f_3(N) + s^5[\tfrac{22}{5}g_5(N) + \tfrac{8}{3}h_5(N)] \\ & - s^7[\tfrac{2336}{315}g_7(N) + \tfrac{176}{45}h_7(N) + \tfrac{64}{27}k_7(N)] \\ & + s^9[\tfrac{7096}{567}g_9(N) + \tfrac{1168}{189}h_9(N) + \tfrac{242}{45}k_9(N) + \tfrac{88}{27}j_9(N) + \tfrac{160}{81}q_9(N)] \\ & - s^{11}\left[\frac{1{,}095{,}064}{51{,}975}g_{11}(N) + \frac{28{,}384}{2835}h_{11}(N) + \frac{12{,}848}{1575}k_{11}(N)\right. \\ & \left. + \tfrac{4672}{945}j_{11}(N) + \tfrac{968}{225}q_{11}(N) + \tfrac{352}{135}m_{11}(N) + \tfrac{128}{81}n_{11}(N)\right] + \dots \end{aligned} \quad (2.6)$$

Here N is the boundary-layer variable based upon the distance n normal to the surface (Schlichting's η). From this, using Tifford's values for the functions $f_1, f_3 \dots n_{11}$, we find the local coefficient of skin friction as

$$\begin{aligned} c_f \equiv \tau/\tfrac{1}{2}\rho U_r^2 = 2R^{-\frac{1}{2}}(1{\cdot}2325877s - 1{\cdot}9318595s^3 + 3{\cdot}1105082s^5 - 5{\cdot}028922s^7 \\ + 8{\cdot}14109s^9 - 13{\cdot}18662s^{11} + \dots). \end{aligned} \quad (2.7)$$

Here $R = U_r L/\nu$ is the Reynolds number based upon the nose radius of the parabola. The first term is Hiemenz's result.

On the basis of the six numerical coefficients available, it is possible to make reasonable conjectures regarding the convergence of the Blasius series. The (unsigned) ratios of successive coefficients in (2.7) are

$$0{\cdot}6380, \quad 0{\cdot}6211, \quad 0{\cdot}6185, \quad 0{\cdot}6178, \quad 0{\cdot}6174, \dots$$

It seems likely that these approach $(\frac{1}{4}\pi)^2 = 0{\cdot}6169$, so that the radius of convergence is $s = \frac{1}{4}\pi$. Thus the convergence of the Blasius series is the same as that of the underlying expansion (2.5c) for the inviscid surface speed.

2.2. *Transformation of Blasius series for skin friction*

We see that the Blasius series is limited not by a physical singularity in the boundary layer, but by a mathematical singularity lying on the imaginary axis in the plane of s regarded as a complex variable. Under these circumstances it is possible to enlarge the range of convergence for positive real s by a change of variable.

One possibility that suggests itself is to recast the series in powers of the parabolic co-ordinate ξ, because it is the natural co-ordinate for the problem. This is equivalent to expanding in powers of the abscissa x, because on the body $\xi^2 = 2x$. Using (2.4) we find that this gives

$$\tfrac{1}{2}R^{\frac{1}{2}}c_f = 1{\cdot}2325877\xi - 1{\cdot}7264282\xi^3 + 2{\cdot}113764\xi^5 - 2{\cdot}441925\xi^7 + 2{\cdot}73149\xi^9 - 2{\cdot}99343\xi^{11} + \ldots. \tag{2.8}$$

The ratios of successive coefficients are now

0·7140, 0·8168, 0·8656, 0·8940, 0·9125, ...

and it seems likely that these approach unity. This means that the series converges for ξ less than unity, which is in accord with the previous observation of singularities at $\xi = \pm i$. In terms of s, however, the radius of convergence has been almost doubled, because according to (2.4) $\xi = 1$ corresponds to $s = 1{\cdot}15$ (whereas $\xi = \pm i$ corresponds to only $s = 0{\cdot}62$).

The singularity that limits convergence now lies at -1 in the complex plane of ξ^2 (which is the relevant variable, since the expansion proceeds by alternate powers of ξ). Thus, as often happens in applied mathematics, an expansion that has physical significance only for positive real values of the argument is restricted by a singularity elsewhere in the complex plane, often on the negative real axis.† Under these circumstances, it may be possible to further enlarge the range of convergence by applying an Euler transformation (see, for example, Bellman 1955). This suggests recasting the series in powers of the new variable

$$z = \xi^2/(1+\xi^2). \tag{2.9}$$

An odd function of ξ must first be extracted from the series (2.8), however, because the skin friction is an odd function whereas z is even. Far downstream on a parabola the skin friction evidently approaches that for a flat plate, because the nose radius becomes negligible relative to the dimensions of interest. This suggests applying the Euler transformation to ξc_f, which gives

$$(\tfrac{1}{2}R_x)^{\frac{1}{2}}\,c_f = 1{\cdot}2325877z - 0{\cdot}4938405z^2 - 0{\cdot}106505z^3 - 0{\cdot}047331z^4 - 0{\cdot}02675z^5 - 0{\cdot}0172z^6 - \ldots. \tag{2.10}$$

Here $R_x = U_r\,x/\nu$ is the Reynolds number based on the abscissa x measured from the leading edge.

We suggest that in this form the series converges to the correct result everywhere on the parabola. The most severe test of this conjecture is at $z = 1$ (corresponding to $\xi = s = x = \infty$) where the skin friction should reach the known

† An example is Goldstein's expansion of the Oseen drag of a sphere in powers of Reynolds number (Goldstein 1938, p. 492). It has been recast as a rational fraction by Shanks (1955), who thereby uncovered an error in Goldstein's last coefficient. Examination of the roots of the denominator in Shanks's expression suggests that convergence of the original series is limited by a singularity at -1 in the complex plane of the parameter $\frac{1}{4}R$ natural to Oseen theory.

value for the semi-infinite flat plate. There the first six terms of (2.10) decrease more rapidly than $1/n^2$, suggesting that the series converges faster than $\Sigma 1/n^2 = \frac{1}{6}\pi^2$. (I am indebted to D. Clutter for this remark.) The successive partial sums are

$$R_x^{\frac{1}{2}} c_f = 1{\cdot}7431, \quad 1{\cdot}0447, \quad 0{\cdot}8941, \quad 0{\cdot}8272, \quad 0{\cdot}7894, \quad 0{\cdot}7651, \quad \ldots. \tag{2.11}$$

It does not appear implausible that these converge to the value for the flat plate, namely 0·6641. This conjecture may be supported in a variety of ways. For example, plotting the nth partial sum S_n versus $1/n$ and extrapolating to the

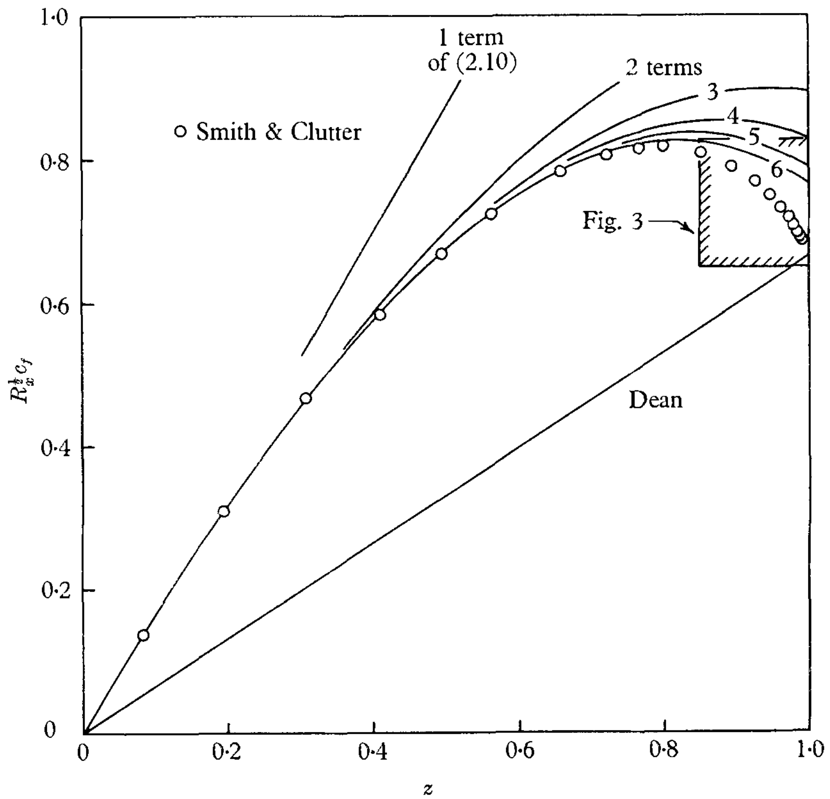

FIGURE 2. Distribution of skin friction over parabola according to first-order boundary-layer theory.

origin gives a value close to that of Blasius. We want to systematize this extrapolation for later use. A simple method is to fit a polynomial in $1/n$ to the last few partial sums. With three terms the sum is predicted to be

$$S = 18S_6 - 25S_5 + 8S_4, \tag{2.12}$$

and the use of any more elaborate scheme is scarcely justified. Applied to (2.11) this yields 0·6554, which differs from the Blasius value by 1·3 % (whereas the sixth partial sum is in error by 15 %).

Figures 2 and 3 show the variation of skin friction along the parabola as given by successive partial sums of our modified Blasius series (2.10). Also shown is an estimate due to Dean (1954), who suggests that Blasius's solution for the flat plate, rewritten in parabolic rather than Cartesian co-ordinates, may be a good approximation everywhere; it is seen to be considerably in error.

Smith & Clutter (1963) have developed a practical numerical procedure for solving the incompressible laminar boundary-layer equations, and have applied

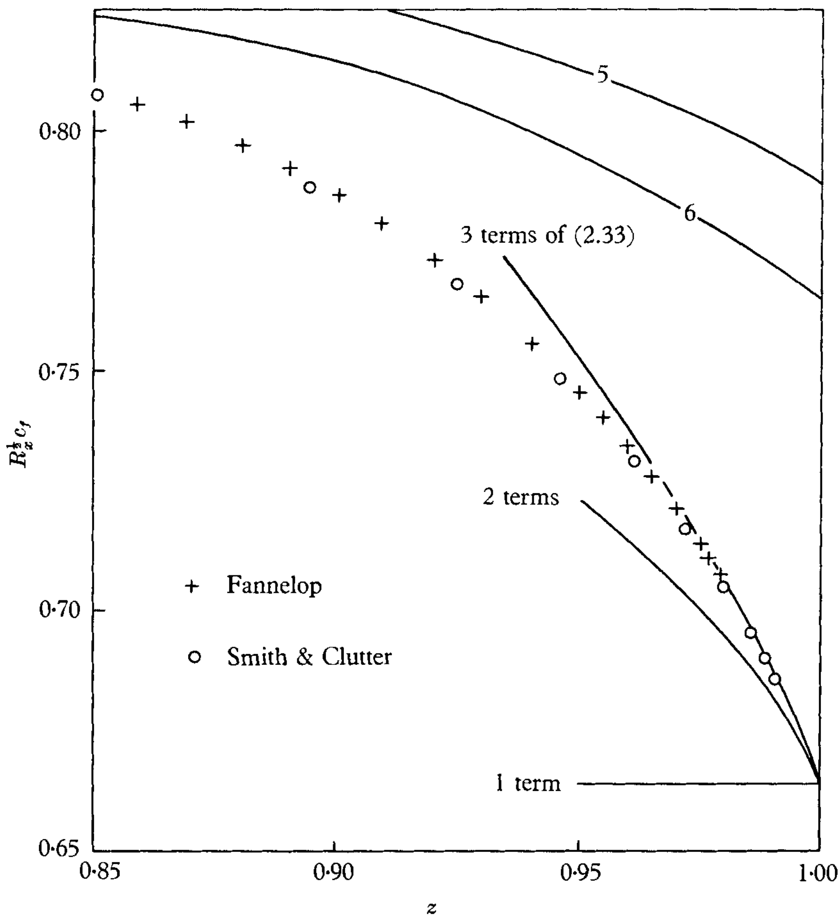

FIGURE 3. Detail of figure 2 far downstream.

it to the parabolic cylinder. Their values of skin friction are compared with the results of the present series in table 1 and in figures 2 and 3. Values beyond $x = 2$ are extracted from an unpublished computation kindly supplied by Smith & Clutter, and believed by them to have four-place accuracy. The general agreement tends to confirm the accuracy of both methods.

Recently, in unpublished work, T. Fannelop has modified the numerical procedure developed by Flügge-Lotz & Blottner (1962) for solving the compressible boundary-layer equations, and has calculated for the present problem the values listed in table 2 and plotted in figure 3. On the basis of convergence tests he believes the error to be no more than a few digits in the fourth place. The two numerical computations do in fact agree to this order over the nose of the

$x = \frac{1}{2}\xi^2$ (nose radii)	$z = \xi^2/(1+\xi^2)$	5 terms of (2.10)	6 terms of (2.10)	Numerical (Smith & Clutter)
0·045	0·0826	0·139080	0·139080	0·13908
0·120	0·1935	0·31002	0·31002	0·3101
0·223	0·3084	0·46608	0·46606	0·4661
0·348	0·4104	0·58498	0·58486	0·5849
0·491	0·4955	0·6687	0·6684	0·6683
0·648	0·5645	0·7254	0·7246	0·7242
0·9655	0·6588	0·7849	0·7829	0·7812
1·2975	0·7218	0·8121	0·8087	0·8050
1·644	0·7668	0·8250	0·8200	0·8138
2·00	0·8000	0·8306	0·8242	0·8152
2·85	0·8507	0·833	0·824	0·8074
4·25	0·8947	0·828	0·816	0·7889
6·15	0·9248	0·821	0·806	0·7683
8·775	0·9461	0·814	0·797	0·7487
12·365	0·9611	0·808	0·789	0·7315
17·25	0·9718	0·804	0·783	0·7170
23·96	0·9796	0·800	0·779	0·7052
33·015	0·9851	0·797	0·775	0·6958
40·87	0·9879	0·796	0·773	0·6905
50·55	0·9902	0·795	0·772	0·6861

TABLE 1. Skin friction on parabola according to first-order boundary-layer theory; values of $R_x^{\frac{1}{2}} c_f$.

z	$R_x^{\frac{1}{2}} c_f$ (Fannelop)	z	$R_x^{\frac{1}{2}} c_f$ (Fannelop)
0·85903	0·80555	0·95512	0·74051
0·86931	0·80190	0·96004	0·73467
0·88095	0·79699	0·96498	0·72841
0·89075	0·79217	0·97012	0·72144
0·90099	0·78640	0·97499	0·71435
0·90951	0·78100	0·97703	0·71122
0·92052	0·77313	0·97912	0·70791
0·93014	0·76536	—	—
0·94058	0·75586	—	—
0·95034	0·74587	—	—

TABLE 2. Skin friction on parabola according to first-order boundary-layer theory.

parabola. Farther downstream, however, Fannelop's values lie consistently higher than those of Smith & Clutter by 0·2 or 0·3 %.

2.3. *Comparison with Görtler's series*

Görtler (1957) has proposed a new series that is expected to be superior to the Blasius series because the variables are adapted to the particular problem at hand. For the parabola, Görtler's variables $\bar{\xi}$, $\bar{U}$, $\bar{g}$, and $\bar{\beta}$ are found to be given in terms of our ξ by

$$\left.\begin{aligned} \bar{\xi} = \tfrac{1}{2}\xi^2, \quad \bar{U} = \xi(1+\xi^2)^{-\frac{1}{2}}, \quad \bar{g} = (1+\xi^2)^{-\frac{1}{2}}, \\ \bar{\beta}(\bar{\xi}) = (1+2\bar{\xi})^{-1}. \end{aligned}\right\} \tag{2.13a}$$

Using these relations together with Görtler's equation (119), and numerical values of the universal functions taken from the abridged tables (Görtler 1955), gives

$$\tfrac{1}{2}(1+\xi^2)\,R^{\frac{1}{2}}c_f = 1{\cdot}232587\xi - 0{\cdot}493840\xi^3 + 0{\cdot}387335\xi^5 - 0{\cdot}328162\xi^7 + 0{\cdot}289567\xi^9 - 0{\cdot}262037\xi^{11} + \ldots \quad (2.13b)$$

This differs from our intermediate result (2.8) only by the extraction of a factor $(1+\xi^2)$, and comparing the two series gives complete agreement, within the accuracy available, between the numerical coefficients derived from the two different sources.

If the radius of convergence is unity for (2.8), then it is also unity here, which seems to be plausible. Thus it appears that Görtler's new series, though indeed somewhat superior to the classical Blasius series, can for the parabola be further significantly improved by an Euler transformation. It is interesting to ask whether a general theory analogous to Görtler's might be developed that for the parabola would provide the infinite radius of convergence of the present *ad hoc* method.

2.4. *Stream function and displacement thickness*

In view of the success of the Euler transformation for the skin friction, we apply it now to the complete stream function in the boundary layer. We work henceforth only with parabolic co-ordinates, for which an appropriate magnified boundary-layer variable H is introduced in place of η by setting

$$H = R^{\frac{1}{2}}(\eta - 1). \quad (2.14)$$

Using (2.1) shows that this is related to the N of Part 1 and (2.6) by

$$N = (1+\xi^2)^{\frac{1}{2}} H. \quad (2.15)$$

In the flat-plate solution, which should hold asymptotically far downstream, the stream function ψ_1 in the boundary layer is proportional to ξ. This suggests that the Euler transformation is to be applied to ψ_1/ξ. Thus with the aid of (2.4) and (2.15) we transform the Blasius series (2.6) into

$$\psi_1/\xi = f_1(H) + (\tfrac{1}{6}f_1 + \tfrac{1}{2}Hf_1' - \tfrac{8}{3}f_3)z + (\tfrac{17}{120}f_1 + \tfrac{11}{24}Hf_1' + \tfrac{1}{8}H^2f_1'' - 4f_3 - \tfrac{4}{3}Hf_3' + \tfrac{22}{5}g_5 + \tfrac{8}{3}h_5)\,z^2 + \ldots \quad (2.16)$$

and anticipate that this series converges over the entire parabola. Here terms up to z^5 are known, but have been omitted in view of their complexity, and because we do not need the complete stream function in the boundary layer but only its displacement thickness.

The displacement effect of the boundary layer depends upon the behaviour of ψ_1 for large values of the argument N or H. The asymptotic forms of the universal functions can be extracted from Tifford's tables and substituting them into (2.16) gives, in the notation of (I, 3.27*b*),

$$-\Psi_2(s,0) = \lim_{N\to\infty}(N\psi_{1N} - \psi_1) = \lim_{H\to\infty}(H\psi_{1H} - \psi_1) = \xi(0{\cdot}647900 + 0{\cdot}183914z + 0{\cdot}09465z^2 + 0{\cdot}05866z^3 + 0{\cdot}0399z^4 + 0{\cdot}0286z^5 + \ldots). \quad (2.17)$$

As a further test of the conjecture that (2.16) converges everywhere on the parabola, consider the displacement effect infinitely far downstream (at $z = 1$). Successive partial sums of (2.17) give for the displacement constant β_1 (cf. (II, 4.5)).

$$\beta_1 = 0{\cdot}6479, \quad 0{\cdot}8318, \quad 0{\cdot}9265, \quad 0{\cdot}9851, \quad 1{\cdot}0251, \quad 1{\cdot}0537, \quad \ldots. \tag{2.18}$$

Applying (2.12) to the last three values yields 1·2199, which differs by only 0·3 % from the known value of 1·21677 for the flat plate.

2.5. *Inverse Blasius series*

The representation of the first-order boundary-layer solution by the modified Blasius series (2.16) appears to be suitable for our purposes. However, it is of interest to examine in more detail the behaviour of the solution far downstream. We consider a supplementary expansion for large rather than small distances from the nose, which may be termed an 'inverse Blasius series'.

The first step would be to expand the inviscid surface speed (2.3) asymptotically for large s, which gives

$$U_1(s,0) \sim 1 - \tfrac{1}{4}s^{-1} - \tfrac{1}{16}s^{-2}\log 8s + \tfrac{1}{32}s^{-2} + \ldots. \tag{2.19}$$

However, we again expand instead for large ξ, because it is a more natural co-ordinate than s. Then (2.3) shows that the surface speed is analytic at $\xi = \infty$; the logarithms in (2.19) arise simply from geometry. Nevertheless, they serve as a reminder that logarithmic terms are often required in asymptotic expansions of this sort to ensure exponential decay of vorticity through the boundary layer (cf. Stewartson 1957). We therefore assume that the stream function can be expanded for large ξ as

$$\psi_1 \sim \xi f_{\mathrm{I}}(H) + \xi^{-1}\log \xi^2 g_{\mathrm{II}}(H) + \xi^{-1} f_{\mathrm{II}}(H) + \ldots. \tag{2.20}$$

Transforming the first-order boundary-layer problem (I, 4.9) to parabolic co-ordinates using (I, 4.22a) gives

$$(1+\xi^2)\,(\psi_{1HHH} + \psi_{1\xi}\psi_{1HH} - \psi_{1H}\psi_{1\xi H}) + \xi(\psi_{1H}^2 + 1) = 0, \tag{2.21a}$$

$$\psi_1(\xi,0) = \psi_{1H}(\xi,0) = 0, \quad \psi_{1H}(\xi,\infty) = \xi. \tag{2.21b}$$

We have already solved this problem by series for small ξ. For large ξ, substituting the assumed expansion (2.20) and equating like functions of ξ gives for the leading term

$$f_{\mathrm{I}}''' + f_{\mathrm{I}} f_{\mathrm{I}}'' = 0, \quad f_{\mathrm{I}}(0) = f_{\mathrm{I}}'(0) = 0, \quad f_{\mathrm{I}}'(\infty) = 1. \tag{2.22a}$$

As anticipated, this is the problem (II, 4.4) for the semi-infinite plate, with the Falkner–Skan normalization, so that

$$f_{\mathrm{I}}''(0) = \alpha_1 = 0{\cdot}469600, \quad f_{\mathrm{I}}(H) \sim H - \beta_1, \quad \beta_1 = 1{\cdot}21677,$$
$$f_{\mathrm{I}}''(H) \sim \gamma_1 \exp[-\tfrac{1}{2}(H-\beta_1)^2], \quad \gamma_1 = 0{\cdot}33054. \tag{2.22b}$$

For the second term, involving the logarithm of ξ, one finds the homogeneous problem

$$g_{\mathrm{II}}''' + f_{\mathrm{I}} g_{\mathrm{II}}'' + 2f_{\mathrm{I}}' g_{\mathrm{II}}' - f_{\mathrm{I}}'' g_{\mathrm{II}} = 0, \quad g_{\mathrm{II}}(0) = g_{\mathrm{II}}'(0) = g_{\mathrm{II}}'(\infty) = 0. \tag{2.23}$$

The solution would vanish if it were unique; but Alden (1948) showed in his pioneering attack on the third approximation for the flat plate that $(Hf_I' - f_I)$ is an eigensolution, so that

$$g_{II}(H) = A_1(Hf_I' - f_I), \tag{2.24}$$

where A_1 is a constant to be determined. Then for the third term in (2.20) the problem becomes

$$f_{II}''' + f_I f_{II}'' + 2f_I' f_{II}' - f_I'' f_{II} = f_I'^2 - 1 + 2A_1 f_I f_I'', \tag{2.25a}$$

$$f_{II}(0) = f_{II}'(0) = f_{II}'(\infty) = 0. \tag{2.25b}$$

This equation, with a different right-hand side (and different normalization), was encountered by Alden. His analysis shows that the behaviour of the solution for small and large H is given by

$$f_{II}''(0) = C_1 f_I''(0) = C_1\alpha_1, \tag{2.26a}$$

$$f_{II}(H) \sim \beta_1 C_1 + a + b\left[\frac{1}{H-\beta_1} + \frac{2}{(H-\beta_1)^2} + \dots\right] + \exp, \tag{2.26b}$$

where 'exp' stands for exponentially small terms. Here C_1 is the coefficient of the eigensolution $(Hf_I' - f_I)$ that appears also as a part of f_{II}, and a and b are determinable constants. An unpublished analysis of S. Kaplun yields, in the present problem,

$$b = \int_0^\infty f_I(1 - f_I'^2)\,dH - A_1. \tag{2.26c}$$

Now the bracket in (2.26*b*) would contribute first-order vorticity that decays only algebraically through the boundary layer. To insure exponential decay, its coefficient b must vanish. Such a requirement is familiar from the Falkner–Skan solutions for the boundary layer in a retarded stream, where Hartree (1937) required exponential decay in order to single out a unique solution. Thus the logarithmic term in (2.20) plays an essential role. A rough numerical computation of the integral in (2.26*c*) yields $A_1 = 0{\cdot}601$.

L. Devan has pointed out to the author a simpler derivation of this result. Multiplying (2.25*a*) by f_I and integrating, using (2.22*a*), gives

$$f_I f_{II}'' + (f_I^2 - f_I')f_{II}' + f_I'' f_{II} = \log f_I'' + \tfrac{1}{2}f_I^2 f_I' + \tfrac{1}{2}f_I f_I'' - \tfrac{1}{4}f_I'^2 + A_1(f_I'^2 - 2f_I f_I'') + k_1. \tag{2.27}$$

Evaluating this at $H = 0$ using the boundary conditions gives

$$k_1 = -\log f_I''(0) = -\log\alpha_1;$$

and then evaluating it again at $H = \infty$, using (2.22*b*) and assuming that f_{II}' decays exponentially, yields

$$A_1 = \tfrac{1}{4} + \log(\alpha_1/\gamma_1) = 0{\cdot}60115. \tag{2.28}$$

Here we have taken advantage of the accurate numerical value

$$\log(\alpha_1/\gamma_1) = 0{\cdot}35115$$

that was computed by Spence (1960) in a different problem.

The coefficient C_1 of the eigensolution remains unknown within the framework of the present asymptotic analysis. Physically, this indeterminacy represents ignorance of the location of the effective origin of the abscissa for the basic flat-plate solution. The same difficulty is encountered in the third-order boundary-layer solution for the flat plate (Goldstein 1960, Imai 1957), and in other boundary-layer problems (e.g. Stewartson 1957, 1958; Traugott 1962).

The eigensolution just discussed is only the first of an infinite sequence (Stewartson 1957). The higher-order eigensolutions have no such simple physical interpretation as the first, because they involve non-integral powers of ξ. Libby & Fox (1962) have calculated the next four exponents as $-1{\cdot}887$, $-2{\cdot}818$, $-3{\cdot}8$, and $-4{\cdot}74$. Thus the next two terms in the inverse expansion (2.20) are eigensolutions of the form $\xi^{-1\cdot 887}f_{\mathrm{III}}(H)$ and $\xi^{-2\cdot 818}f_{\mathrm{IV}}(H)$, each introducing another unknown coefficient. These are followed by conventional terms of order $\xi^{-3}\log^2\xi$, $\xi^{-3}\log\xi$, and ξ^{-3}, then by two more eigensolutions with exponents $-3{\cdot}8$ and $-4{\cdot}74$, and so on.

2.6. *Joining of direct and inverse Blasius series*

In the present problem it is possible to evaluate approximately the constant C_1 for the first eigensolution, thanks to the infinite radius of convergence of the series from the nose. The direct and inverse series can be joined by a process that is neither the precise matching of the method of inner and outer expansions (Part 1), nor the crude numerical patching to be employed later. It is almost the process of analytical continuation, except that our inverse series is not analytic. We find it convenient to join the two series for the skin friction.

According to (I, 4.13) the inverse Blasius series (2.20) gives for the skin-friction coefficient

$$(\tfrac{1}{2}R_x)^{\frac{1}{2}}\,c_f \sim 0{\cdot}469600\,[1+0{\cdot}60115\xi^{-2}\log\xi^2+(C_1-1)\,\xi^{-2}]. \qquad (2.29a)$$

For purposes of comparison we recast this in terms of the variable $z = \xi^2/(1+\xi^2)$, and transfer the logarithmic term to the left-hand side

$$(\tfrac{1}{2}R_x)^{\frac{1}{2}}\,c_f-0{\cdot}28230(1-z)\log(1-z)^{-1} \sim 0{\cdot}469600[1+(C_1-1)(1-z)]. \qquad (2{\cdot}29b)$$

We now evaluate this same expression using the modified Blasius series (2.10). The expansion

$$(1-z)\log(1-z)^{-1} = z-\tfrac{1}{2}z^2-\tfrac{1}{6}z^3-\tfrac{1}{12}z^4-\ldots \qquad (2.30)$$

converges for $0 \leqslant z \leqslant 1$, so that the desired result

$$(\tfrac{1}{2}R_x)^{\frac{1}{2}}\,c_f-0{\cdot}28230\,(1-z)\log(1-z)^{-1} = 0{\cdot}950288z-0{\cdot}352690z^2$$
$$-0{\cdot}059455z^3-0{\cdot}023806z^4-0{\cdot}012640z^5-0{\cdot}00775z^6-\ldots \qquad (2.31a)$$

converges up to $z = 1$. Expanding it formally for small $(1-z)$ gives

$$(\tfrac{1}{2}R_x)^{\frac{1}{2}}\,c_f-0{\cdot}28230\,(1-z)\log(1-z)^{-1}\sim (0{\cdot}950288-0{\cdot}352690-0{\cdot}059455$$
$$-0{\cdot}023806-0{\cdot}012640-0{\cdot}00775-\ldots)$$
$$+(-0{\cdot}950288+0{\cdot}705380+0{\cdot}178365+0{\cdot}095224+0{\cdot}063200$$
$$+0{\cdot}04650+\ldots)\,(1-z)+\ldots. \qquad (2.31b)$$

This result is only asymptotic, because although the two subseries in parentheses appear to converge, that for the next term in $(1-z)^2$ would not until a term in

$(1-z)^2 \log(1-z)^{-1}$ was extracted. The first subseries is again presumably approaching the flat-plate value. We need the value of the second, whose convergence is more marginal. Applying (2.12) to the last three partial sums yields 0·42, and we adopt this estimate while recognizing that the second digit is uncertain.

Comparing (2.29*b*) and (2.31*b*) then determines the missing coefficient C_1 according to

$$C_1 = 1+(-0{\cdot}950288+0{\cdot}705380+0{\cdot}178365+0{\cdot}095224+0{\cdot}063200 + 0{\cdot}04650+\ldots)/0{\cdot}469600 \approx 1{\cdot}89. \quad (2.32)$$

The skin friction is now given by the inverse Blasius series as

$$R_x^{\frac{1}{2}} c_f \sim 0{\cdot}66411+0{\cdot}39923(1-z)\log(1-z)^{-1}+0{\cdot}59(1-z)+\ldots. \quad (2.33)$$

The results of using one, two and three terms of this series are shown in figure 3.

The constant C_1 can also be evaluated approximately by patching the skin friction far downstream with the numerical results. Equating (2.29*b*) to the values of Smith & Clutter at the last six points listed in table 1 gives $C_1 = 1{\cdot}66$, 1·68, 1·69, 1·67, 1·64, and 1·59. Patching with the results of Fannelop at the last six points in table 2 gives $C_1 = 1{\cdot}72$, 1·75, 1·78, 1·81, 1·82, and 1·83. Together with the value estimated by joining the two series, these results indicate that C_1 lies somewhere between 1·6 and 1·9.

3. Second-order boundary-layer solution

Of the second-order effects enumerated in Part 1, only (longitudinal) curvature and displacement are present here. In order to apply existing results for those effects at the nose, we need only calculate the change in inviscid surface speed induced there by the first-order displacement thickness.

3.1. *Flow due to displacement thickness*

The boundary layer alters the outer inviscid flow by the order of its displacement thickness, so that the stream function has the outer expansion

$$\Psi_1 + R^{-\frac{1}{2}}\Psi_2 + \ldots. \quad \text{(I, 3.2)}$$

Here Ψ_2 must satisfy Laplace's equation according to (I, 3.12), give vanishing velocity far upstream, and assume at the body the value in (2.17).

This potential problem can be solved formally by inspection. The first term in (2.17), a multiple of ξ, is a harmonic function as it stands, with proper decay at infinity, and can therefore be taken over directly. The subsequent terms are not harmonic, and must be replaced by appropriate harmonic functions that reduce to them at $\eta = 1$. The required functions are combinations of ξ with multipoles at the origin: the real (imaginary) parts of odd (even) negative powers of $(\xi + i\eta)$. Thus the flow due to displacement thickness is found to be represented formally by

$$\Psi_2(\xi,\eta) = -0{\cdot}647900\,\xi - 0{\cdot}183914\left(\xi - \frac{\xi}{\xi^2+\eta^2}\right) - 0{\cdot}09465\left[\xi - 2\frac{\xi}{\xi^2+\eta^2} + \frac{\xi\eta}{(\xi^2+\eta^2)^2}\right] - \ldots. \quad (3.1)$$

The subsequent three terms are known but have been omitted here because of their increasing complexity; we need only the corresponding surface speed, which is found from (I, 2.10) as

$$U_2(s,0) = (1+\xi^2)^{-\frac{1}{2}}\Psi_{2\eta}(\xi,1) = -\xi(1+\xi^2)^{-\frac{3}{2}}[0{\cdot}367828+0{\cdot}094655(1+4z) + 0{\cdot}05866(\tfrac{3}{4}+2z+6z^2)+0{\cdot}03996(\tfrac{5}{8}+\tfrac{3}{2}z+3z^2+8z^3) + 0{\cdot}0286(\tfrac{35}{64}+\tfrac{5}{4}z+\tfrac{9}{4}z^2+4z^3+10z^4)+\ldots]. \quad (3.2)$$

This series appears to diverge for ξ greater than unity ($z > \frac{1}{2}$). Some reflexion suggests recasting it to leave only a factor $\xi/(1+\xi^2)$ outside the bracket, and it is reassuring to find that the result then appears to converge for all ξ. However, (3.2) suffices as it stands to give the first non-zero term in the expansion (II, 3.10) for the displacement speed near the nose as

$$U_{21} = -(0{\cdot}367828+0{\cdot}094655+0{\cdot}04399+0{\cdot}02497+0{\cdot}0157+\ldots) \approx -0{\cdot}61. \quad (3.3)$$

Here successive terms again decrease faster than $1/n^2$, and the estimate given for the sum is the result of applying (2.12).

3.2. *Second-order skin friction and heat transfer at nose*

The value of U_{21} was the missing element needed to complete the second-order boundary-layer solution near the nose. Substituting it into (II, 3.36), together with $U_{11} = 1$ from (2.5*c*), gives for the coefficient of skin friction

$$\tfrac{1}{2}R^{\frac{1}{2}}c_f = (1{\cdot}232588-3{\cdot}03R^{-\frac{1}{2}})\,s/L+O(s^3,R^{-1}). \quad (3.4)$$

In this section only, all symbols denote actual dimensional quantities, so that s/L is the distance from the stagnation point measured in nose radii. Curvature contributes $-1{\cdot}91$ to the second coefficient and displacement $-1{\cdot}12$; thus both second-order effects act to reduce the skin friction near the nose, curvature being twice as effective as displacement. Higher powers of s could be found by continuing the second-order Blasius series, whereas terms of relative order R^{-1} would come from the third-order boundary-layer solution.

Similarly, for Prandtl number 0·7 the low-speed heat transfer at the stagnation point is found from (II, 3.37) as

$$q/k = (U_r/\nu)^{\frac{1}{2}}(T_{w0}-T_0)[0{\cdot}495867-0{\cdot}279R^{-\frac{1}{2}}+O(R^{-1})]. \quad (3.5)$$

In this case displacement contributes $-0{\cdot}151$ to the second-order term and curvature only $-0{\cdot}128$.

3.3. *Discussion*

Our solution for the parabola has no direct application to the puzzle of the semi-infinite flat plate, where near the leading edge the flow is unknown, and far downstream undetermined constants begin to appear in the third approximation of boundary-layer theory. We cannot let our nose Reynolds number R tend to zero, because the analysis is asymptotic for large R, valid only if a well-defined boundary layer exists beginning at the stagnation point.

Nevertheless, it is remarkable that second-order effects reduce the skin friction near the nose. This trend is in the direction of the Oseen approximation, for which Wilkinson (1955) finds the skin friction on the parabola to be

$$c_f = 2\pi^{-\frac{1}{2}}R_x\xi^2/(1+\xi^2) = 2R^{\frac{1}{2}}[0{\cdot}798s+O(s^3)]. \quad (3.6)$$

Near the stagnation point this gives about two-thirds of the boundary-layer value of Hiemenz. Far downstream, on the other hand, it exceeds the Blasius value by 70 %; the Oseen approximation does not apply to a semi-infinite plane body even if the nose Reynolds number is small.

The displacement effect was calculated here by something of a *tour de force*, using the Euler transformation. One cannot expect equal success for all bodies. It might be possible, however, to accept the approximation of the Kármán–Pohlhausen integral method; and it would be useful to compare that result with the present solution.

A primitive version of this work, in which the displacement effect was ignored, was presented in 1956 (Van Dyke 1957). The analysis was revised during the summer of 1960 at the Lockheed Missiles and Space Company (Van Dyke 1960). The present paper was prepared under Air Force Office of Scientific Research Grant AFOSR-96-63. The author is indebted for advice and help to D. Clutter, L. Devan, T. Fannelop, S. Goldstein, P. A. Lagerstrom, R. Mark, N. Rott, and A. M. O. Smith.

Corrections to previous parts

Part 1: Page 165, equation (2.12) should read: $\cos\theta = -\kappa^{-1}r_0''$.

Page 167, two lines below (3.11*c*): replace B_1 by B_1'.

Page 170, line 7 should read: $m = 1, p = 2$.

Part 2: Page 484, equation (2.23): replace 0·48296 by 0·15831 and 0·50194 by 0·17079.

Page 484, line 3: Kemp's solution is found to agree with ours when put in the same terms. His parameter A is not to be identified with the gradient U_{11} of inviscid surface speed, but depends upon the vorticity parameter Ω of (2.27) according to $U_{11} = A(1-\beta_1\Omega)$. This is easily seen by calculating the outer flow that matches Kemp's solution according to the principle (I, 3.24). Consequently his results reduce for small Ω to those of our § 2.4.

Page 485, equation (2.28): replace 0·96592 by 0·31661 and 1·00388 by 0·34158.

Page 489, equation (3.36): close square bracket after U_{11}^2).

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